Notes on 2.2

We want to determine fixed points of functions:
def. p is a fixed point of g if $g(p) = p$

Fixed point algorithms can be made into root finding algorithms (and vice-versa)

Theorem 2.3

1. if $g \in C([a, b])$ and g takes vakes in $[a, b]$, then g has at least one fixed point.
2. if g has a derivative whose absolute value is strictly bounded by 1, then it has a unique fixed point.

Proof part 1 of 2.3.
really clever:

1. let $h(x) = g(x) - x$
2. if $h(a) = 0$ or $h(b) = 0$ then we have a fixed point
3. otherwise, since $g(a) > a$ and $g(b) < b$, we can apply intermediate value theorem to h

Proof part 2 of 2.3
if there are two fixed points, then they both are on the line $y = x$ (draw a picture). But by the mean value theorem, we get a point where the derivative is 1, this is a contradiction.

Fixed point iteration
to determine fixed point for a continuous function:

Algorithm

1. guess p_0
2. while $g(p_i) \neq p_i$
   a. let $p_{i+1} = g(p_i)$

(literally just keep plugging in hoping things work out)

Theorem 2.4 (Fixed-Point Theorem)
if $g \in C([a, b])$ with absolute value of derivative strictly less than 1 on (a, b), then the fixed point algorithm converges to a unique fixed point.
Proof.

1. $|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi)||p_{n-1} - p| \le k|p_{n-1} - p|$
2. $|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le \cdots \le k^n|p_0 - p| \to 0$ since $k < 1$

Corollary 2.5
$$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0|$$
Proof.

1. $|p_n - p| = \lim_{m\to\infty} |p_n - p_m|$
2. $|p_n - p_m| \le \sum_{i=0}^{m-n-1} |p_{n+i+1} - p_{n+i}| \le \sum_{i=0}^{m-n-1} k^{n+i}|p_1 - p_0| = k^n|p_1 - p_0| \sum_{i=0}^{m-n-1} k^i$
3. $\lim_{m\to\infty} |p_n - p_m| \le \lim_{m\to\infty} k^n|p_1 - p_0| \sum_{i=0}^{m-n-1} k^i = k^n|p_1 - p_0| \sum_{i=0}^{\infty} k^i = k^n|p_1 - p_0| \frac{1}{1-k}$